Another proof of the mean ergodic theorem.

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Von Neumann's "mean ergodic theorem" will be given a new short proof in the following generalized form, established by the author some three years ago, together with other theorems of the same kind, concerning function spaces L^p.[1])

Theorem. *Let T be a bounded linear transformation in Hilbert space, for which $M_T \leq 1$, i. e. $|Tf| \leq |f|$ for any element f. Let f_1 be an element, let $f_n = T^{n-1} f_1$ and let φ_n be the arithmetic mean of $f_1, \ldots, f_n$. Then the sequence φ_n converges to a limit φ. More generally,*

$$\varphi_{m,n} = \frac{1}{n-m} \sum_{m+1}^{n} f_i \to \varphi \qquad (n-m \to \infty).$$

Proofs of the above theorem and of its extensions to more or less general spaces have been given on repeated occasions by several authors. In particular, it is with the proof given by GARRETT BIRKHOFF[2]) that our new argument has some features in common. Neither of them depends on weak compactness in spite of using minimal methods; in addition, both hold for any uniformly convex space. For sake of convenience, we write down the present argument for HILBERT space only; but it clearly may be read so as to embody the general case.

Birkhoff's starting-point is the infimum of the norms $|\varphi_n|$; instead, we consider the infimum μ of the norm $|g|$ on the whole

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[1]) F. RIESZ, Some mean ergodic theorems, *Proceedings London Math. Society*, **13** (1938), pp. 274–278.

[2]) G. BIRKHOFF, The mean ergodic theorem, *Duke Math. Journal*, **5** (1939), pp. 19–20; *cf.* also L. ALAOGLU and G. BIRKHOFF, General ergodic theorems, *Annals of Math.*, **41** (1940), pp. 293–309.

convex set G of elements g of the type

$$g=\sum_1^{\nu} c_i f_i \quad \left(\nu \text{ arbitrary, } c_i \geqq 0,\ \sum_1^{\nu} c_i = 1\right). \tag{1}$$

Clearly, the means φ_n and $\varphi_{m,n}$ belong to the set G.

In order to prove our theorem, we propose to show first that the sequence $\{\varphi_n\}$ and more generally, any sequence $\{\varphi_{m,n}\}$ with $n-m\to\infty$ is a minimizing sequence, *i. e.*, $|\varphi_n|\to\mu$ and $|\varphi_{m,n}|\to\mu$. As

$$|\varphi_{m,n}|=|T^m\varphi_{n-m}|\leqq|\varphi_{n-m}|,$$

the case of $\varphi_{m,n}$ in fact reduces to that of φ_n and all we have to show is that for any $\varepsilon>0$,

$$|\varphi_N|<\mu+\varepsilon \tag{2}$$

for large N. To this purpose, let us take any element g of the type (1) with

$$|g|<\mu+\frac{\varepsilon}{2} \tag{3}$$

and observe that the arithmetic mean ψ_N of $g, Tg, \ldots, T^{N-1}g$, when expressed in terms of f_1 and its transforms f_k, differs from φ_N, for $N>\nu-1$, but in its first and last terms, of the form $a_k f_k$ with $0\leqq a_k\leqq 1/N$, $2(\nu-1)$ in number, and so clearly

$$|\psi_N-\varphi_N|\leqq\frac{2(\nu-1)}{N}|f_1|<\frac{\varepsilon}{2} \tag{4}$$

for large N. As, in addition, on account of the hypothesis $M_T\leqq 1$, (3) holds with $T^k g$ instead of g and thus for the mean ψ_n too, the inequality (2) readily follows from (4). Therefore, $\{\varphi_n\}$ and $\{\varphi_{m,n}\}$ are minimizing sequences.

Now, to finish our argument, we only have to recall the well known fact stating that for a convex set in HILBERT space, minimizing sequences are always convergent[3]). So, in particular, this is the case for the sequences $\{\varphi_{m,n}\}$ which proves the theorem.

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[3]) See *e. g.* B. DE SZ. NAGY, On the set of positive functions in L_2, *Annals of Math.*, **39** (1938), pp. 1—13, in particular p. 5, footnote.